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Increasing (decreasing) pairwise soft connected in soft bitopological ordered spaces

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Abstract. In this paper, we introduce the notion of IPS (DPS)-soft sets based on the soft bitopological ordered space $(X, \tau_1, \tau_2, E, \lesssim)$ and study some of its properties. Based on this notion we introduce the notions of IP (DP)- soft connected (disconnected) spaces and study some of their characterizations and properties. Also, we study the connected of IP (DP)-soft sets by using the soft space $(X, \tau_{12}, E, \lesssim)$. Some examples have given to support these concepts.

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1. Introduction

In 1965, Nachbin [1] proposed the concept of topological ordered spaces which add partial order relations to topological structures as a generalization of topological spaces. McCartan [2] went on to utilize monotone neighborhoods in order to study ordered separation axioms. In order to deal with the vagueness and uncertainty of real-life problems, various mathematical tools have been developed such as fuzzy sets, intuitionistic fuzzy sets, rough sets, and vague sets. One such tool, soft sets, was introduced by Molodtsov [3] in 1999 and has since been developed and applied to decision-making problems, algebraic structures, and topological spaces.

Şenel [4] presented the soft topology generated by L-soft sets. Additionally, in 2016, Şenel [5] proposed a new approach to Hausdorff space theory via the soft sets. Şenel and Çağman [6] introduced soft topological subspaces. Also they [7] explored soft closed sets on soft bitopological space. In 2020, Şenel et al. [8] investigated distance and similarity measures for octahedron sets proposed by Lee et al. [9].

El-Sheikh et al. [10] extended the idea of soft topological spaces by introducing supra soft topological spaces. In a similar vein, Ittanagi [11] proposed the concept of soft bitopological spaces, which are defined over an initial universal set and incorporate a fixed set of parameters. Kandil et al. [12, 13] provided some structures on soft bitopological spaces and defined some basic notions such as pairwise open and closed soft sets, pairwise soft closure, interior, kernel operators, and related topics. They also studied pairwise soft continuous mappings and open and closed soft mappings between two soft bitopological spaces. Additionally, they studied the concept of soft connectedness in soft bitopological spaces, the concepts of pairwise separated soft sets, pairwise soft connected (disconnected) spaces, and pairwise connected soft sets.

El-Shafei et al. [14, 15] proposed two innovative forms of soft relations, established the concepts of monotone soft sets and increasing (descending) soft operators, revealing crucial insights into their fundamental properties. Moreover, they introduced the notion of soft topological ordered spaces and formulated ordered soft separation axioms.

Additionally, El-Sheikh et al. [16] introduced the concept of soft bitopological ordered spaces, which includes increasing (decreasing) pairwise open (closed) soft sets, as well as the notions of increasing (decreasing) total (partial) pairwise soft neighborhoods and increasing (decreasing) pairwise open soft neighborhoods. They also studied the relationships between these concepts, including the increasing (decreasing) pairwise soft closure (interior).

The purpose of this article is to introduce and study the concept of soft connectedness in soft bitopological ordered spaces. We study the concepts of increasing (decreasing) pairwise separated soft sets, increasing (decreasing) pairwise soft connected (disconnected) spaces and increasing (decreasing) pairwise connected soft sets. The rest of this paper is organized as follows. In Section 2, we introduced briefly the notions of soft set, soft topology, soft bitopological spaces, soft bitopological ordered spaces, soft mapping and some related topics. In Section 3, we introduce the notion of increasing (decreasing) pairwise separated soft sets and give some characterizations of these soft sets. In Section 4, we introduce the notions of increasing (decreasing) pairwise soft connected (disconnected) spaces and investigate some of their properties. In Section 5, we give the concept of $IP(DP)$-connected soft sets and some related properties are studied.

2. Preliminaries

This section provides a brief overview of key concepts and relevant results from the fields of soft sets, soft topological spaces, soft bitopological spaces, and soft topological ordered spaces, which will be used in this paper. For more detailed information on these topics, please refer to [10, 12, 13, 14, 16, 17, 18, 19, 20].

Definition 2.1 ([3])**.** Let X be a universe set and let E be a fixed set of parameters. If $G_E : E \to 2^X$ is a function, then an ordered pair (G, E) is called a *soft set*, where 2^X is the power set of X. The set of all soft sets over X is denoted by $P(X)^E$.

Definition 2.2 ([21])**.** Let $F_E,\ G_E \in P(X)^E$.

(i) F_E is said to be a *null soft set*, denoted by Φ, if $F(e) = \varnothing\ \forall e \in E$.
(ii) F_E is called an *absolute soft set*, denoted by X_E, if $F(e) = X\ \forall e \in E$.
(iii) F_E is called a *soft subset* of G_E, denoted by $F_E \sqsubseteq G_E$, if $F(e) \subseteq G(e)\ \forall e \in E$.

(iv) F_E and G_E are said to be *equal*, denoted by $F_E = G_E$, if $F_E \sqsubseteq G_E$ and $G_E \sqsubseteq F_E$.

(v) The *union* of F_E and G_E is a soft set H_E, defined as: $H(e) = F(e) \cup G(e)\ \forall e \in E$. We write $H_E = F_E \sqcup G_E$.

(vi) The *intersection* of F_E and G_E is a soft set H_E, defined as: $H(e) = F(e) \cap G(e)\ \forall e \in E$. We write $H_E = F_E \sqcap G_E$.

(vii) The *difference* of F_E and G_E is a soft set H_E, defined as: $H(e) = F(e) - G(e)\ \forall e \in E$. We write $H_E = F_E - G_E$.

(viii) The *complement* of F_E, denoted by F_E^c, defined by: $F^c(e) = (F(e))^c\ \forall e \in E$.

Definition 2.3 ([22, 23])**.** A soft set $H_E : E \to 2^X$ defined as $H(\alpha) = \{x\}$ if $e = \alpha$ and $H(e) = \varnothing$ if $e \in E - \{\alpha\}$ is called a *soft point* and denoted by x^α. The collection of all soft points over X is denoted by $Sp(X)^E$. A soft point x^α is said to *belong to a soft set* G_E, denoted by $x^\alpha \widehat{\in} G_E$, if $x^\alpha(\alpha) \subseteq G(\alpha)$ for each $\alpha \in E$.

Definition 2.4 ([22])**.** Let $\phi : X \to Y$ and $\psi : E \to K$ be two mappings. Then the mapping $\phi_\psi : P(X)^E \to P(Y)^K$ is called a *soft mapping from X to Y*. Let $G_E \in P(X)^E$ and let $F_K \in P(Y)^K$.

(i) The *soft image of $G_E \in P(X)^E$ under ϕ_ψ*, denoted by $\phi_\psi(G_E)$, is a soft set over Y defined as follows: for each $k \in K$,

$$\phi_\psi(G_E)(k) = \bigcup_{\alpha \in \psi^{-1}(k)} G(\alpha) \text{ if } \psi^{-1}(k) \neq \varnothing \text{ and } \phi_\psi(G_E)(k) = \varnothing \text{ otherwise.}$$

(ii) The *soft inverse image of F_K under ϕ_ψ*, denoted by $\phi_\psi^{-1}(F_K)$, is a soft set over X defined as follows: for each $e \in E$,

$$\phi_\psi^{-1}(F_K)(e) = \phi^{-1}(F(\psi(e))).$$

Definition 2.5 ([24])**.** A soft mapping $\phi_\psi : P(X)^E \to P(Y)^K$ is called a *soft surjective* (resp. *injective*) *mapping*, if ϕ and ψ are surjective (resp. injective) mappings, respectively.

Proposition 2.6 ([14])**.** *The following two results hold for a soft mapping $\phi_\psi : P(X)^E \to P(Y)^K$.*

(1) *The image of each soft point is soft point.*
(2) *If ϕ_ψ is bijective, then the inverse image of each soft point is soft point.*

Theorem 2.7 ([22])**.** *Let $G_E^i \in P(X)^E$ and $H_K^i \in P(Y)^K$ for all $i \in J$, where J is an index set. Then, for a soft mapping $\phi_\psi : P(X)^E \to P(Y)^K$, the following conditions are satisfied:*

(1) *if $G_E^1 \sqsubseteq G_E^2$, then $\phi_\psi(G_E^1) \sqsubseteq \phi_\psi(G_E^2)$,*
(2) *if $H_K^1 \sqsubseteq H_K^2$, then $\phi_\psi^{-1}(H_K^1) \sqsubseteq \phi_\psi^{-1}(H_K^2)$,*
(3) $\phi_\psi(\sqcup_{i \in J}(G_E^i)) = \sqcup_{i \in J}(\phi_\psi(G_E^i))$,
(4) $\phi_\psi^{-1}(\sqcup_{i \in J}(H_K^i)) = \sqcup_{i \in J}(\phi_\psi^{-1}(H_K^i))$,
(5) $\phi_\psi^{-1}(\sqcap_{i \in J}(H_K^i)) = \sqcap_{i \in J}(\phi_\psi^{-1}(H_K^i))$,
(6) $\phi_\psi(\phi_\psi^{-1}(H_K)) \sqsubseteq H_K$,
(7) $\phi_\psi^{-1}(Y_K) = X_E$, $\phi_\psi^{-1}(\widehat{\phi}_K) = \widehat{\phi}_E$ *and* $\phi_\psi(\widehat{\phi}_E) = \widehat{\phi}_K$.

Proposition 2.8 ([22]). *Let $\phi_\psi : P(X)^E \to P(Y)^K$ be a soft map and let $G_E \in P(X)^E$, $F_K \in P(X)^E$. Then we have the following results:*

(1) $G_E \sqsubseteq \phi_\psi^{-1}(\phi_\psi(G_E))$, *and the equality holds if ϕ_ψ is injective.*
(2) $\phi_\psi(\phi_\psi^{-1}(F_K)) \sqsubseteq F_K$, *and the equality holds if ϕ_ψ is surjective.*
(3) $\phi_\psi^{-1}(F_K^c) = (\phi_\psi^{-1}(F_K))^c$ *for any $F_K \in P(Y)^K$.*

Definition 2.9 ([3, 14]). For a soft set H_E over X and an element $x \in X$, we say $x \in H_E$ if $x \in H(e)$ for every $e \in E$ and $x \notin H_E$ if $x \notin H(e)$ for some $e \in E$. We say $x \Subset H_E$ if $x \in H(e)$ for some $e \in E$ and $x \not\Subset H_E$ if $x \notin H(e)$ for every $e \in E$. The notations $\in, \notin, \Subset$ and $\not\Subset$ are respectively read as belong, non-belong, partial belong and total non-belong relations.

Definition 2.10 ([25]). A *soft topology* τ on X is a collection of soft sets over X under E that satisfy the following conditions:
(i) the null soft set and the absolute soft set are included in τ,
(ii) the union of any collection of soft sets in τ is also in τ,
(iii) the intersection of any two soft sets in τ is also in τ.
The triple (X, τ, E) is called a *soft topological space over* X. Each member of τ is referred to as a *soft open set* and its relative complement is called a *soft closed set*.

Definition 2.11 ([24]). A soft subset N_E of a soft topological space (X, τ, E) is called a *soft neighborhood* of $x \in X$, if there exists a soft open set G_E such that $x \in G_E \sqsubseteq N_E$.

Definition 2.12 ([22]). A soft map $\phi_\psi : (X, \tau, E) \to (Y, \eta, K)$ is said to be *soft continuous*, if the inverse image of each soft open subset of (Y, η, K) is a soft open subset of (X, τ, E).

Definition 2.13 ([11, 12]). A quadrable system (X, η_1, η_2, E) is called a *soft bitopological space*, where η_1 and η_2 represent soft topologies on the set X with a fixed set of parameters E. Let H_E be a soft set over a soft bitopological space (X, η_1, η_2, E). Then H_E is called a:

(i) *pairwise open soft* (briefly, PO-soft) *set*, if there exists an η_1-open soft set H_E^1 and an η_2-open soft set H_E^2 such that $H_E = H_E^1 \sqcup H_E^2$,
(ii) *pairwise closed soft* (briefly, PC-soft) *set*, if H_E^c is a PO-soft set.

Furthermore, the family of all PO-soft sets, denoted by η_{12}, forms a supra soft topological space associated with the soft bitopological space (X, η_1, η_2, E).

Definition 2.14 ([13]). Let (X, τ_1, τ_2, E) be a soft bitopological space and let G_E, H_E are non-null soft sets over X. Then G_E and H_E are said to be *pairwise separated* (briefly, P-separated) soft sets, if $Scl_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Scl_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$.

Definition 2.15 ([13]). Let (X, τ_1, τ_2, E) be a soft bitopological space. Then P-separated soft sets G_E and H_E in (X, τ_1, τ_2, E) are said to be a *pairwise soft separation* of X (briefly, P-soft separation), if $X_E = G_E \sqcup H_E$. In this case, we say that X_E *has an P-soft separation.*

Definition 2.16 ([13]). A soft bitopological space (X, τ_1, τ_2, E) is said to be a:

(i) *pairwise soft disconnected space* (briefly, P-soft disconnected), if X_E has a P-soft separation.

(ii) *pairwise soft connected space* (briefly, P-soft connected), if it is not P-soft disconnected, i.e., X_E has not an P-soft separation.

Definition 2.17 ([26])**.** A binary relation $\lesssim$ on a set X is called a *partial order relation* on X, if it is reflexive, anti-symmetric, and transitive. The *equality relation* on X, denoted by $\blacktriangle$, is defined as $\{(x, x) : x \in X\}$.

Definition 2.18 ([1])**.** A triple $(X, \tau, \lesssim)$ is called a *topological ordered space*, if (X, τ) is a topological space and $(X, \lesssim)$ is a partially ordered set.

Definition 2.19 ([14])**.** A triple $(X, E, \lesssim)$ is called a *partially ordered soft space*, if $\lesssim$ is a partial order relation on the set X.

(i) An *increasing soft operator* $i : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ is defined as follows: for each $H_E \in P(X)^E$,

$$i(H_E)(\alpha) = iH(\alpha) = \{x \in X : \delta \lesssim x \text{ for some } \delta \in H(\alpha)\}.$$

(ii) A *decreasing soft operator* $d : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ is defined as follows: for each $H_E \in P(X)^E$,

$$d(H_E)(\alpha) = dH(\alpha) = \{x \in X : x \lesssim \delta \text{ for some } \delta \in H(\alpha)\}.$$

(iii) A soft subset H_E of the partially ordered soft space $(X, E, \lesssim)$ is said to be *increasing* (resp. *decreasing*), if $H_E = i(H_E)$ (resp. $H_E = d(H_E)$).

Proposition 2.20 ([14])**.** *Let $i : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ and $d : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ be increasing and decreasing soft operators, and let H_E and G_E be two soft sets in $(X, E, \lesssim)$. Then*

(1) $i(\widehat{\phi}_E) = \widehat{\phi}_E$ *and* $d(\widehat{\phi}_E) = \widehat{\phi}_E$,
(2) $H_E \sqsubseteq i(H_E)$ *and* $H_E \sqsubseteq d(H_E)$,
(3) $i(i(H_E)) = i(H_E)$ *and* $d(d(H_E)) = d(H_E)$,
(4) $i[H_E \sqcup G_E] = i(H_E) \sqcup i(G_E)$*and* $d[H_E \sqcup G_E] = d(H_E) \sqcup d(G_E)$.

Definition 2.21 ([14])**.** (i) A quadrable system $(X, \tau, E, \lesssim)$ called a *soft topological ordered space* (briefly, $STOS$), if (X, τ, E) is a soft topological space and $(X, E, \lesssim)$ is a partially ordered soft space.

(ii) A soft set H_E in a soft topological ordered space $(X, \tau, E, \lesssim)$ is called an *increasing* (resp. *decreasing*) *open soft set*, if it is soft open and increasing (resp. decreasing).

Definition 2.22 ([14])**.** A soft subset N_E of an STOS $(X, \tau, E, \lesssim)$ is called an *increasing* (resp. a *decreasing*) *soft neighborhood* of $x \in X$, if N_E is a soft neighborhood of x and increasing (resp. decreasing).

Definition 2.23 ([20])**.** A quadrable system A $(X, \tau_1, \tau_2, \lesssim)$ is called a *bitopological ordered space* (briefly, *bto*), if $(X, \lesssim)$ is a partially ordered space and (X, τ_1, τ_2) is a *bts*.

Definition 2.24 ([16])**.** A quinary system $(X, \tau_1, \tau_2, E, \lesssim)$ is called a *soft bitopological ordered space* (briefly, SBTOS), if the following conditions hold:

(i) (X, τ_1, τ_2, E) is a soft bitopological space,
(ii) $(X, E, \lesssim)$ is a partially ordered soft space.

Definition 2.25 ([16]). Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$. A soft set M_E over X is said to be:

(i) an *increasing pairwise open soft* (briefly, IPO-soft) *set*, if $M_E = M_E^1 \sqcup M_E^2$, $M_E^\beta \in \tau_\beta$ and M_E^β is increasing, $\beta = 1,\ 2$,

(ii) a *decreasing pairwise open soft* (briefly, DPO-soft) *set*, if $M_E = M_E^1 \sqcup M_E^2$, $M_E^\beta \in \tau_\beta$ and M_E^β is decreasing, $\beta = 1,\ 2$,

(iii) an *increasing pairwise closed soft* (briefly, IPC-soft) *set*, if $M_E = M_E^1 \sqcap M_E^2$, $M_E^\beta \in \tau_\beta^c$ and M_E^β is increasing, $\beta = 1,\ 2$,

(iv) a decreasing pairwise closed soft (briefly, DPO-soft) *set*, if $M_E = M_E^1 \sqcap M_E^2$, $M_E^\beta \in \tau_\beta^c$ and M_E^β is decreasing, $\beta = 1,\ 2$.

Definition 2.26 ([16]). Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$.

(i) The *increasing pairwise soft closure* of G_E, denoted by $Icl^s_{12}(G_E)$, is the intersection of all increasing pairwise closed soft sets including G_E, i.e.,

$$Icl^s_{12}(G_E) = \sqcap\{F_E : F_E \text{ is } IPC\text{-soft set, } G_E \sqsubseteq F_E\}.$$

(ii) The *decreasing pairwise soft closure* of G_E, denoted by $Dcl^s_{12}(G_E))$, is the intersection of all decreasing pairwise closed soft sets including G_E, i.e.,

$$Dcl^s_{12}(G_E) = \sqcap\{K_E : K_E \text{ is } DPC\text{-soft set, } G_E \sqsubseteq K_E\}.$$

It is clear that $Icl^s_{12}(G_E)(Dcl^s_{12}(G_E))$ is the smallest IPC (resp. DPC)-soft set including G_E.

(iii) The *increasing pairwise soft interior* of G_E, denoted by $Iint^s_{12}(G_E))$, is the union of all increasing pairwise open soft sets embodied in G_E, i.e.,

$$Iint^s_{12}(G_E) = \sqcup\{O_E : O_E \text{ is } IPO\text{-soft set, } O_E \sqsubseteq G_E\}.$$

(vi) The*decreasing pairwise soft interior* of G_E, denoted by $Dint^s_{12}(G_E))$, is the union of all decreasing pairwise open soft sets embodied in G_E, i.e.,

$$Dint^s_{12}(G_E) = \sqcup\{M_E : M_E \text{ is } DPO\text{-soft set, } M_E \sqsubseteq G_E\}.$$

It is obvious that $Iint^s_{12}(G_E)(Dint^s_{12}(G_E))$ is the largest IPO (resp. DPO)-soft set embodied in G_E.

Corollary 2.27 ([16]). *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$. Then $Icl^s_{\tau_{12}}(G_E) = Icl^s_{\tau_1}(G_E) \sqcap Icl^s_{\tau_2}(G_E)$.*

Theorem 2.28 ([16]). *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \lesssim_1) \to (Y, \eta_1, \eta_2, K, \lesssim_2)$ be a soft mapping. The following statements are equivalent:*

(1) *ϕ_ψ is ISP-continuous,*

(2) *$\phi_\psi(Icl^s_{12}(G_E)) \sqsubseteq cl^s_{12}(\phi_\psi(G_E))$ for any $G_E \in P(X)^E$,*

(3) *$Icl^s_{12}(\phi_\psi^{-1}(F_K)) \sqsubseteq \phi_\psi^{-1}(cl^s_{12}(F_K))$ for any $F_K \in P(Y)^K$,*

(4) *for any PC-soft subset M_K of $(Y, \eta_1, \eta_2, K, \lesssim_2)$, $\phi_\psi^{-1}(M_K)$ is DPC-soft subset of $(X, \tau_1, \tau_2, E, \lesssim_1)$.*

Definition 2.29 ([16]). A soft set G_E in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be a *total pairwise soft neighborhood* of $x \in X$, if there is a $PO-$soft set H_E such that $x \in H_E \sqsubseteq G_E$.

Definition 2.30 ([16]). A soft set W_E in an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is called an *increasing pairwise soft neighborhood* (briefly, IPS-nbd) (resp. a *decreasing pairwise soft neighborhood* (briefly, DPS-nbd)) of $x^e \widehat{\in} X_E$, if there exists a $PO-$soft set H_E such that $x^e \widehat{\in} H_E \sqsubseteq W_E$ and W_E is increasing (resp. decreasing).

Definition 2.31 ([16])**.** A soft set G_E in an $SBTOS$ $(X,\tau_1,\tau_2,E,\lesssim_1)$ is called:
(i) an *increasing total pairwise soft neighborhood* (briefly, $ITPS$-nbd) of $x\in X$, if G_E is a total pairwise soft neighborhood of x and increasing,
(ii) a *decreasing total pairwise soft neighborhood* (briefly, $DTPS$-nbd) of $x\in X$, if G_E is a total pairwise soft neighborhood of x and decreasing.

Definition 2.32 ([16])**.** An $SBTOS$ $(X,\tau_1,\tau_2,E,\lesssim)$ is said to be:
(i) *lower pairwise soft $T_1^\bullet$-ordered* (briefly, $LPST_1^\bullet$-ordered), if for any distinct points x and y in X such that $x\not\lesssim y$, there exists an $ITPS$-nbd G_E of x such that $y\notin G_E$,
(ii)*upper pairwise soft $T_1^\bullet$-ordered* (briefly, $UPST_1^\bullet$-ordered), if for any distinct points x and y in X such that $x\not\lesssim y$,there exists a $DTPS$-nbd G_E of y such that $x\notin G_E$,
(iii) *lower pairwise soft SST_1-ordered* (briefly, $LPSST_1$-ordered), if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1}\not\lesssim y^{e_2}$, there exists an IPS-nbd W_E of x^{e_1} such that $y^{e_2}\widehat{\notin}W_E$,
(iv) *upper pairwise soft SST_1-ordered* (briefly, $UPSST_1$-ordered), if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1}\not\lesssim y^{e_2}$, there exists a DPS-nbd W_E of y^{e_2} such that $x^{e_1}\widehat{\notin}W_E$.

Corollary 2.33 ([16])**.** *For an SBTOS $(X,\tau_1,\tau_2,E,\lesssim)$, the family of all IPO-soft and DPO-soft sets forms an increasing supra soft topology, denoted by τ_{12}^{IP}, and decreasing supra soft topology, denoted by τ_{12}^{DP}, respectively on X. It is also mentioned that the decreasing supra soft topology of complements of sets in τ_{12}^{IP} is equivalent to the increasing supra soft topology of complements of sets in τ_{12}^{DP} and vice versa. In fact,*
$$\tau_{12}^{IP}=\{M_E: M_E=M_E^1\sqcup M_E^2, M_E^\beta\in\tau_\beta \text{ and increasing, } \beta=1,2\},$$
$$\tau_{12}^{DP}=\{N_E: N_E=N_E^1\sqcup N_E^2, N_E^\beta\in\tau_\beta \text{ and decreasing, } \beta=1,2\}.$$
However, $\tau_{12}^{cIP}=\{H_E^c: H_E\in\tau_{12}^{DP}\},\ \tau_{12}^{cDP}=\{O_E^c: O_E\in\tau_{12}^{IP}\}$.

Definition 2.34 ([16])**.** Let $Y\subseteq X$ and $(X,\tau_1,\tau_2,E,\lesssim)$ be an $SBTOS$. Then $(Y,\tau_{1Y},\tau_{2Y},E,\lesssim_Y)$ is called soft $bi-$ordered subspace of $(X,\tau_1,\tau_2,E,\lesssim)$, provided that $(Y,\tau_{1Y},\tau_{2Y},E)$ is soft bitopological subspace of (X,τ_1,τ_2,E), where $\lesssim_Y$ is a partial order relation on Y.

3. Increasing (Decreasing) pairwise separated soft sets

Definition 3.1. Let $(X,\tau_1,\tau_2,\lesssim)$ be a *bto*. A subset A of X is said to be:
(i) an *increasing pairwise open set* (briefly, IPO-set), if $A=A^1\cup A^2, A^\beta\in\tau_\beta$ and A^β is increasing, $\beta=1,\ 2$,
(ii) a *decreasing pairwise open set* (briefly, DPO-set), if $A=A^1\cup A^2, A^\beta\in\tau_\beta$ and A^β is decreasing, $\beta=1,\ 2$,
(iii) an *increasing pairwise closed set* (briefly, IPC-set), if $A=A^1\cup A^2, A^\beta\in\tau_\beta^c$ and A^β is increasing, $\beta=1,\ 2$,
(iv) a *decreasing pairwise closed set* (briefly, DPC-set), if $A=A^1\cup A^2, A^\beta\in\tau_\beta^c$ and A^β is decreasing, $\beta=1,\ 2$.

Definition 3.2. Let $(X,\tau_1,\tau_2,\lesssim)$ be a *bto* and let $A\in 2^X$.

(i) The *increasing pairwise closure* of A, denoted by $Icl_{12}(A)$, is the intersection of all IPC-sets containing A, i.e., $Icl_{12}(A) = \cap\{B : B \text{ is } IPC\text{-set}, A \subseteq B\}$.

(ii) The *decreasing pairwise closure* of A, denoted by $Dcl_{12}(A)$, is the intersection of all DPC-sets containing A i.e., $Dcl_{12}(A) = \cap\{K : K \text{ is } DPC\text{-set}, A \subseteq K\}$.
It is obvious that $Icl_{12}(A)$ (resp. $Dcl_{12}(A)$) is the smallest IPC (resp. DPC)-sets containing A.

(iii) The *increasing pairwise soft interior* of A, denoted by $Iint_{12}(A)$, is the union of all IPO-sets contained in A, i.e., $Iint_{12}(A) = \cup\{O : O \text{ is } IPO\text{-set}, O \subseteq A\}$.

(iv) The *decreasing pairwise soft interior* of A, denoted by $Dint_{12}(A)$, is the union of all DPO-soft sets contained in A, i.e., $Dint_{12}(A) = \cup\{G : G \text{ is } DPO\text{-set}, G \subseteq A\}$.
It is clear that $Iint_{12}(A)$ (resp. $Dint_{12}(A)$) is the largest IPO (resp. DPO)-sets contained in A

Definition 3.3. Let $(X, \tau_1, \tau_2, \lesssim)$ be a *bto* and let $A,\ B$ be non-null subsets of X. Then A and B are said to be:

(i) an *increasing pairwise separated sets* (briefly, IPS-sets), if $Icl_{12}(A) \cap B = \varnothing$ and $Icl_{12}(B) \cap A = \varnothing$,

(ii) a *decreasing pairwise separated sets* (briefly, DPS-sets), if $Dcl_{12}(A) \cap B = \varnothing$ and $Dcl_{12}(B) \cap A = \varnothing$.

Definition 3.4. Let $(X, \tau_1, \tau_2, \lesssim)$ be a *bto* and let $A,\ B$ be IPS (resp. DPS)-sets in X. Then A ans B are said to be an *increasing* (resp. *decreasing*) *pairwise separation* of X (briefly, IP (resp. DP)-separation), if $X = A \cup B$. In this case, we say that X *has an* IP (resp. DP)*-separation.*

Definition 3.5. A *bto* $(X, \tau_1, \tau_2, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise disconnected space* (briefly, IP (resp. DP)-disconnected), if X has an IP (resp. DP)-separation. Otherwise, $(X, \tau_1, \tau_2, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise connected space* (briefly, IP (resp. DP)-connected), i. e., A *sbo* $(X, \tau_1, \tau_2, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise connected*, if X has not an IP (resp. DP)-separation.

Definition 3.6. Let $(X, \tau, E, \lesssim)$ be an $STOS$ and let $G_E,\ H_E$ be non-null soft sets over X. Then G_E and H_E are said to be:

(i) *increasing soft separated sets* (briefly, ISS-sets), if $Icl^s_\tau(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_\tau(H_E) \sqcap G_E = \widehat{\phi}_E$,

(ii) *decreasing soft separated sets* (briefly, DSS-sets), if $Dcl^s_\tau(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Dcl^s_\tau(H_E) \sqcap G_E = \widehat{\phi}_E$.

Definition 3.7. Let $(X, \tau, E, \lesssim)$ be an $STOS$. Then ISS (resp. DSS)-sets G_E and H_E in X are said to be an *increasing* (resp. a *decreasing*) *soft separation* (briefly, IS (resp. DS)-separation) of X, if $X_E = G_E \sqcup H_E$. In this case, we say that X *has an* IS (resp. a DS)*-separation.*

Definition 3.8. An $STOS$ $(X, \tau, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *soft disconnected space* (briefly, IS (resp. DS)-disconnected), if X has an IS (resp. a DS)-separation. Otherwise, $(X, \tau, E, \lesssim)$ is said to be an *increasing* (resp. *decreasing*) *soft connected space* (briefly, IS (resp. DS)-connected), i.e., an STOS $(X, \tau, E, \lesssim)$ is said to be *increasing* (resp. *decreasing*) *soft connected*, if X has not an IS (resp. a DS)-separation.

Definition 3.9. Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and let $G_E,\ H_E$ are non-null soft sets over X. Then G_E and H_E are said to be:

(i) *increasing pairwise separated soft sets* (briefly, IPS-soft sets), if $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$,

(ii) *decreasing pairwise separated soft sets* (briefly, DPS-soft sets), if $Dcl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Dcl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$.

Proposition 3.10. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$. Then every IPS* (resp. *DPS)-soft sets are disjoint soft sets.*

Proof. Straightforward. □

Remark 3.11. The converse of Proposition 3.10 may not be true as shown by the following example.

Example 3.12. Let $E = \{e_1, e_2\}$ be a set of parameters, $\lesssim = \blacktriangle \cup \{(x, w)\}$ be a partial order relation on $X = \{x, y, z, w\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E, H_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, M_E, N_E\}$, where

$G_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\},\ H_E = \{(e_1, \{y, z\}), (e_2, \{x, y\})\},$
$M_E = \{(e_1, \{w\}), (e_2, \{y, w\})\},\ N_E = \{(e_1, \{x, w\}), (e_2, X)\}.$

It is easy to verify that:

$$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E, M_E, N_E, P^1_E, P^2_E\},$$

where $P^1_E = \{(e_1, \{x, w\}), (e_2, \{y, z, w\})\},\ P^2_E = \{(e_1, \{y, z, w\}), (e_2, \{x, y, w\})\}$.
Then clearly, $\tau^c_{12} = \{\widehat{\phi}_E, X_E, G^c_E, H^c_E, M^c_E, N^c_E, P^{1c}_E, P^{2c}_E\}$,
where $G^c_E = \{(e_1, \{y, z\}), (e_2, \{x, y\})\},\ H^c_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\},$
$M^c_E = \{(e_1, \{x, y, z\}), (e_2, \{x, z\})\},\ N^c_E = \{(e_1, \{y, z\}), (e_2, \varnothing)\},$
$P^{1c}_E = \{(e_1, \{y, z\}), (e_2, \{x\})\},\ P^{2c}_E = \{(e_1, \{x\}), (e_2, \{z\})\}.$

Thus we have

(1) The family of all $IPC-$soft sets are H^c_E and N^c_E,
(2) The family of all $DPC-$soft sets are G^c_E, M^c_E, N^c_E, P^{1c}_E and P^{2c}_E.

Now, consider soft sets F^α_E, $\alpha = 1,\ 2,\ 3$ given by:

$$F^1_E = \{(e_1, \{x\}), (e_2, \{x\})\},\ F^2_E = \{(e_1, \{y\}), (e_2, \varnothing)\},\ F^3_E = \{(e_1, \varnothing), (e_2, \{z\})\}.$$

It is clear that F^2_E, F^3_E are IPS (resp. DPS)-soft sets. Although the soft sets F^1_E and F^3_E are disjoint, we find that they are not IPS-soft sets because

$$Icl^s_{12}(F^3_E) \sqcap F^1_E = \{(e_1, \{x\}), (e_2, \varnothing)\} \neq \widehat{\phi}_E.$$

Also they are not DPS-soft sets because

$$Dcl^s_{12}(F^3_E) \sqcap F^1_E = \{(e_1, \{x\}), (e_2, \varnothing)\} \neq \widehat{\phi}_E.$$

Proposition 3.13. *Every IPS* (resp. *DPS)-soft sets are P-separated soft sets.*

Proof. The proof is given from the fact $cl^s_{12}(G_E) \sqsubseteq Icl^s_{12}(G_E)$. □

The converse of the above Proposition is not true in general.

Example 3.14. From Example 4.3, let $F_E^4 = \{(e_1, \{y\}), (e_2, \{y\})\}$, $F_E^5 = \{(e_1, \{x\}), (e_2, \{w\})\}$. Although the soft sets F_E^4 and F_E^5 are P-separated soft sets, we find that their are not IPS-soft sets because $Icl_{12}^s(F_E^4) \sqcap F_E^5 = \{(e_1, \{x\}), (e_2, \{w\})\} \neq \widehat{\phi}_E$. Also their are not DPS-soft sets because $Dcl_{12}^s(F_E^5) \sqcap F_E^4 = \{(e_1, \{y\}), (e_2, \{y\})\} \neq \widehat{\phi}_E$.

Proposition 3.15. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and let G_E, H_E be non-null soft sets over X.*

(1) *If $Icl_{12}^s(G_E) \sqcap Icl_{12}^s(H_E) = \widehat{\phi}_E$, then G_E and H_E are $IPS-$soft sets.*
(2) *If $Dcl_{12}^s(G_E) \sqcap Dcl_{12}^s(H_E) = \widehat{\phi}_E$, then G_E and H_E are $DPS-$soft sets.*

Proof. Straightforward. □

Note: From Proposition 3.10, 3.13, 3.15, we deduce that the concept of IPS (resp. DPS)-soft sets is a weaker than of the condition of disjoint increasing (resp. decreasing) pairwise soft closure of soft sets, but it is a stronger than of the concept of P-separated soft sets and disjoint soft sets.

Remark 3.16. The converse of Proposition 3.15 may not be true as shown by the following example.

Example 3.17. Let $E = \{e_1, e_2\}$ be the set of parameters, $\lesssim = \blacktriangle \cup \{(y, w)\}$ be a partial order relation on $X = \{x, y, z, w\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E, H_E^1, H_E^2, H_E^3, H_E^4, H_E^5, H_E^6\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, M_E, N_E\}$,
where $G_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\}$.
$H_E^1 = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$, $H_E^2 = \{(e_1, \{x\}), (e_2, \{x\})\}$,
$H_E^3 = \{(e_1, \{x\}), (e_2, \varnothing)\}$, $H_E^4 = \{(e_1, \{x, w\}), (e_2, \{x, z, w\})\}$,
$H_E^5 = \{(e_1, \varnothing), (e_2, \{x\})\}$, $H_E^6 = \{(e_1, \{x, y, z\}), (e_2, \{x, y\})\}$,
$M_E = \{(e_1, \{w\}), (e_2, \{y, w\})\}$, $N_E = \{(e_1, \{x, w\}), (e_2, X)\}$.
It is easy to verify that:

$$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E^1, H_E^2, H_E^3, H_E^4, H_E^5, H_E^6, M_E, N_E, P_E^1, P_E^2, P_E^3, P_E^4, P_E^5, P_E^6\},$$

where $P_E^1 = \{(e_1, \{x, w\}), (e_2, \{y, z, w\})\}$, $P_E^2 = \{(e_1, \{y, z, w\}), (e_2, \{x, y, w\})\}$
$P_E^3 = \{(e_1, \{x, w\}), (e_2, \{x, y, w\})\}$, $P_E^4 = \{(e_1, X), (e_2, \{x, y, w\})\}$,
$P_E^5 = \{(e_1, \{x, w\}), (e_2, \{y, w\})\}$, $P_E^6 = \{(e_1, \{w\}), (e_2, \{x, y, w\})\}$.
Then we have

$$\tau_{12}^c = \{\widehat{\phi}_E, X_E, G_E^c, H_E^{1c}, H_E^{2c}, H_E^{3c}, H_E^{4c}, H_E^{5c}, H_E^{6c}, M_E^c, N_E^c, P_E^{1c}, P_E^{2c}, P_E^{3c}, P_E^{4c}, P_E^{5c}, P_E^{6c}\},$$

where $G_E^c = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$,
$H_E^{1c} = \{(e_1, \{x, w\}), (e_2, \{z, w\})\}$, $H_E^{2c} = \{(e_1, \{y, z, w\}), (e_2, \{y, z, w\})\}$,
$H_E^{3c} = \{(e_1, \{y, z, w\}), (e_2, X)\}$, $H_E^{4c} = \{(e_1, \{y, z\}), (e_2, \{y\})\}$,
$H_E^{5c} = \{(e_1, X), (e_2, \{y, z, w\})\}$, $H_E^{6c} = \{(e_1, \{w\}), (e_2, \{z, w\})\}$,
$M_E^c = \{(e_1, \{x, y, z\}), (e_2, \{x, z\})\}$, $N_E^c = \{(e_1, \{y, z\}), (e_2, \varnothing)\}$,
$P_E^{1c} = \{(e_1, \{y, z\}), (e_2, \{x\})\}$, $P_E^{2c} = \{(e_1, \{x\}), (e_2, \{z\})\}$,
$P_E^{3c} = \{(e_1, \{y, z\}), (e_2, \{z\})\}$, $P_E^{4c} = \{(e_1, \varnothing), (e_2, \{z\})\}$,
$P_E^{5c} = \{(e_1, \{y, z\}), (e_2, \{x, z\})\}$, $P_E^{6c} = \{(e_1, \{x, y, z\}), (e_2, \{z\})\}$.
Thus we have

(1) The family of all IPC-soft sets are H_E^{1c}, H_E^{2c}, H_E^{3c}, H_E^{5c}, H_E^{6c}, P_E^{2c} and P_E^{4c},

(2) The family of all DPC-soft sets are G_E^c, H_E^{2c}, H_E^{3c}, H_E^{5c}, M_E^c, N_E^c, P_E^{1c}, P_E^{2c}, P_E^{3c}, P_E^{4c}, P_E^{5c} and P_E^{6c}.

Now, let $F_E^1 = \{(e_1, \{y, z, w\}), (e_2, \varnothing)\}$, $F_E^2 = \{(e_1, \{x\}), (e_2, \varnothing)\}$. Then we have

$$Icl_{12}^s(F_E^1) = Dcl_{12}^s(F_E^1) = H_E^{2c} = \{(e_1, \{y, z, w\}), (e_2, \{y, z, w\})\}$$

and

$$Icl_{12}^s(F_E^2) = Dcl_{12}^s(F_E^2) = P_E^{2c} = \{(e_1, \{x\}), (e_2, \{z\})\}.$$

It is clear that

$$Icl_{12}^s(F_E^1) \sqcap F_E^2 = \widehat{\phi}_E,\ Icl_{12}^s(F_E^2) \sqcap F_E^1 = \widehat{\phi}_E$$

and

$$Dcl_{12}^s(F_E^1) \sqcap F_E^2 = \widehat{\phi}_E,\ Dcl_{12}^s(F_E^2) \sqcap F_E^1 = \widehat{\phi}_E.$$

Thus F_E^1, F_E^2 are IPS (resp. DPS)-soft sets. But we get

$$Icl_{12}^s(F_E^1) \sqcap Icl_{12}^s(F_E^2) = \{(e_1, \varnothing), (e_2, \{z\})\} \neq \widehat{\phi}_E$$

and

$$Dcl_{12}^s(F_E^1) \sqcap Dcl_{12}^s(F_E^2) = \{(e_1, \varnothing), (e_2, \{z\})\} \neq \widehat{\phi}_E.$$

Proposition 3.18. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and let G_E, H_E be IPO* (resp. *DPO*)*-soft sets. Then G_E, H_E are IPS* (resp. *DPS*)*-soft sets if and only if G_E and H_E are disjoint soft sets.*

Proof. The proof of necessary condition is obvious from Proposition 3.10.

Suppose that G_E and H_E are disjoint IPO-soft sets. Then clearly, $G_E \sqsubseteq H_E^c$, $H_E^c \in \tau_{12}^{IPc}$. It follows that $Icl_{12}^s(G_E) \sqsubseteq H_E^c$ implies $Icl_{12}^s(G_E) \sqcap H_E = \widehat{\phi}_E$. By similar way, we can show that $Icl_{12}^s(H_E) \sqcap G_E = \widehat{\phi}_E$. □

Proposition 3.19. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and let F_E, M_E be IPC* (resp. *DPC*)*-soft sets. Then F_E, M_E are IPS* (resp. *DPS*)*-soft sets if and only if F_E and M_E are disjoint soft sets.*

Proof. Straightforward. □

Theorem 3.20. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$, $Y \subseteq X$ and let G_E, $H_E \sqsubseteq Y_E \sqsubseteq X_E$. If G_E and H_E are IPS* (resp. *DPS*)*-soft sets in $(X, \tau_1, \tau_2, E, \lesssim)$, then they are IPS* (resp. *DPS*)*-soft sets in $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim_Y)$.*

Proof. Suppose G_E and H_E are IPS-soft sets over X. Then we get

$$\begin{aligned} Icl_{12Y}^s(G_E) &= \sqcap\{F_E : F_E \in \tau_{12Y}^{IPc} : G_E \sqsubseteq F_E\} \\ &= \sqcap\{Y_E \sqcap M_E : M_E \in \tau_{12X}^{IPc} : G_E \sqsubseteq M_E\} \\ &= Y_E \sqcap [\sqcap\{M_E : M_E \in \tau_{12X}^{IPc} : G_E \sqsubseteq M_E\}] \\ &= Y_E \sqcap Icl_{12X}^s(G_E). \end{aligned}$$

Thus $Icl_{12Y}^s(G_E) \sqcap H_E = Y_E \sqcap H_E \sqcap Icl_{12X}^s(G_E)$. So we have

$$Icl_{12Y}^s(G_E) \sqcap H_E \sqsubseteq H_E \sqcap Icl_{12X}^s(G_E) = \widehat{\phi}_E.$$

Hence $Icl_{12Y}^s(G_E) \sqcap H_E = \widehat{\phi}_E$. By similar way, we can prove that $Icl_{12Y}^s(H_E) \sqcap G_E = \widehat{\phi}_E$. Therefore G_E, H_E are IPS-soft sets in $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim_Y)$. □

Theorem 3.21. *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be* $LPST_1^\bullet$ (resp. $UPST_1^\bullet$)*-ordered and let* G_E, H_E *be two finite and disjoint increasing* (resp. *decreasing*) *soft sets. Then* G_E *and* H_E *are* IPS (resp. DPS)*-soft sets.*

Proof. Since $(X, \tau_1, \tau_2, E, \lesssim)$ is $LPST_1^\bullet-$ordered, every crisp point is a PC-soft set. Since G_E and H_E are finite soft sets, G_E and H_E are IPC-soft sets. It follows by Proposition 3.19 that G_E and H_E are IPS-soft sets. □

Theorem 3.22. *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be* $UPSST_1$ (resp. $LPSST_1$)*-ordered and let* G_E, H_E *be two finite and disjoint increasing* (resp. *decreasing*) *soft sets. Then* G_E *and* H_E *are* IPS (resp. DPS)*-soft sets.*

Proof. Since $(X, \tau_1, \tau_2, E, \lesssim)$ is $UPSST_1$-ordered, every soft point is a PC-soft set. Since G_E and H_E are finite soft sets, G_E and H_E are IPC-soft sets. It follows by Proposition 3.19 that G_E and H_E are IPS-soft sets. □

Theorem 3.23. *Let* $\phi_\psi : (X, \tau_1, \tau_2, E, \lesssim_1) \to (Y, \eta_1, \eta_2, K, \lesssim_2)$ *be an* ISP (resp. a DSP)*-continuous and soft surjective mapping. If* M_E *and* N_E *are* IPS (resp. DPS)*-soft sets in* $(Y, \eta_1, \eta_2, K, \lesssim_2)$*, then* $\phi_\psi^{-1}(M_E)$ *and* $\phi_\psi^{-1}(N_E)$ *are* IPS (resp. DPS)*-soft sets in* $(X, \tau_1, \tau_2, E, \lesssim_1)$.

Proof. Suppose M_K and N_K are IPS-soft sets in $(Y, \eta_1, \eta_2, K, \lesssim_2)$. Then we have

$$Icl_{12}^s(M_K) \sqcap N_K = \widehat{\phi}_K,\ Icl_{12}^s(N_K) \sqcap M_K = \widehat{\phi}_K.$$

Since ϕ_ψ is an ISP-continuous mapping, by Theorem 2.28, we get

$$Icl_{12}^s[\phi_\psi^{-1}(M_K)] \sqsubseteq \phi_\psi^{-1}[Icl_{12}^s(M_K)].$$

Thus by Theorem 2.7, we have

$$\begin{aligned} Icl_{12}^s[\phi_\psi^{-1}(M_K)] \sqcap \phi_\psi^{-1}(N_K) &\sqsubseteq \phi_\psi^{-1}[Icl_{12}^s(M_K)] \sqcap \phi_\psi^{-1}(N_K) \\ &= \phi_\psi^{-1}[Icl_{12}^s(M_K) \sqcap N_K] \\ &= \phi_\psi^{-1}(\widehat{\phi}_K) \\ &= \widehat{\phi}_E. \end{aligned}$$

So $Icl_{12}^s[\phi_\psi^{-1}(M_K)] \sqcap \phi_\psi^{-1}(N_K) = \widehat{\phi}_E$. Similarly, we can prove that

$$Icl_{12}^s[\phi_\psi^{-1}(N_K)] \sqcap \phi_\psi^{-1}(M_K) = \widehat{\phi}_E.$$

Since ϕ_ψ is soft surjective mapping, $\phi_\psi^{-1}(M_K) \neq \widehat{\phi}_E$ and $\phi_\psi^{-1}(N_K) \neq \widehat{\phi}_E$. Hence $\phi_\psi^{-1}(M_K)$ and $\phi_\psi^{-1}(N_K)$ are IPS-soft sets in $(X, \tau_1, \tau_2, E, \lesssim_1)$. □

In Proposition 3.18 and Theorems 3.20, 3.21, 3.22, 3.23, a similar proof can be given for the case between parentheses.

Theorem 3.24. *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be an* $SBTOS$*. If* G_E *and* H_E *are* IPS (resp. DPS)*-soft sets in* $(X, \tau_1, \tau_2, E, \lesssim)$*, then* $G(e)$ *and* $H(e)$ *are* IPS (resp. DPS)*-sets in* $(X, \tau_1^e, \tau_2^e, \lesssim)$ $\forall e \in E$.

Proof. Suppose G_E and H_E are IPS-soft sets in X and let $e \in E$. Then we have

$$\tau_{12}^{IPe} = \tau_{12}^{IP}(e) = \{G(e) : G_E \in \tau_{12}^{IP}\}.$$

Since $Icl^s_{12}(G_E) = \bigcap\{F_E : F_E \in \tau_{12}^{IPc} : G_E \sqsubseteq F_E\}$, we get

$$Icl^s_{12}(G_E)(e) = \bigcap\{F(e) : F(e) \in \tau_{12}^{IPc}(e) : G(e) \subseteq F(e)\}.$$

Thus $Icl^s_{12}(G_E)(e) = Icl^s_{12}(G(e))$. Now, since G_E and H_E are IPS-soft sets in $(X, \tau_1, \tau_2, E, \lesssim)$, we have

$$Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E \text{ and } Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E.$$

So we get

$$[Icl^s_{12}(G_E) \sqcap H_E](e) = \varnothing \text{ and } [Icl^s_{12}(H_E) \sqcap G_E](e) = \varnothing.$$

Hence we have

$$Icl^s_{12}(G(e)) \cap H(e) = \varnothing \text{ and } Icl^s_{12}(H(e)) \cap G(e) = \varnothing.$$

It follows by Definition 3.3 that, $G(e)$ and $H(e)$ are IPS (resp. DPS)-sets in $(X, \tau_1^e, \tau_2^e, \lesssim)$. □

4. Ordered pairwise soft disconnected (connected) spaces

Definition 4.1. Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$, and let G_E and H_E be IPS (resp. DPS)-soft sets over X. Then G_E and H_E are said to be an *increasing* (resp. *decreasing*) *pairwise soft separation* of X (briefly, IP (resp. DP)-soft separation), if $X_E = G_E \sqcup H_E$. In this case, we say that X_E *has an IP* (resp. *a DP)-soft separation*.

Definition 4.2. An $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise soft disconnected space* (briefly, IP (resp. DP)-soft disconnected), if X_E has an IP (resp. DP)- soft separation. Otherwise, $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise soft connected space* (briefly, IP (resp. DP)-soft connected), i.e., an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise soft connected*, if X_E has not an IP (resp. a DP)-soft separation.

Example 4.3. Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(x, w)\}$ be a partial order relation on $X = \{x, y, z, w\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E, H_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, M_E\}$,
where $G_E = \{(e_1, \{x, w\}), (e_2, \{y, z\})\}$, $H_E = \{(e_1, \{y, z\}), (e_2, \{x, w\})\}$,
$M_E = \{(e_1, \varnothing), (e_2, \{x\})\}$.
It is easy to verify that:
$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E, M_E, N_E\}$, where $N_E = \{(e_1, \{x, w\}), (e_2, \{x, y, z\})\}$.
It is clear that G_E and H_E are form an IP-soft separation and DP-soft separation of X_E. Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP-soft disconnected and DP-soft disconnected space.

Proposition 4.4. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$.*

(1) *If $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E\}$, then $(X, \tau_1, \tau_2, E, \lesssim)$ is IP* (resp. *DP)-soft connected.*

(2) *If $\tau_{12} = P(X)^E$, $\mid X \mid > 1$, then $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP* (resp. *DP)-soft disconnected.*

(3) *If $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E, G_E\}$, then $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP* (resp. *DP)-soft connected.*

Proof. (1) Suppose $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E\}$ and let G_E and H_E be soft sets over X. Then clearly, we have

$$Icl^s_{12}(G_E) = Icl^s_{12}(H_E) = X_E \text{ and } Dcl^s_{12}(G_E) = Dcl^s_{12}(H_E) = X_E.$$

Thus we cannot represented X_E as a union of two IPS (resp. DPS)-soft sets in $(X, \tau_1, \tau_2, E, \lesssim)$. So $(X, \tau_1, \tau_2, E, \lesssim)$ is IP (resp. DP)-soft connected.

(2) Suppose $\tau_{12} = P(X)^E$, $\mid X \mid > 1$. Then every soft set is an IP (resp. DP)-closed soft set. It follows that for every soft point $x^e \widehat{\in} X_E$, we have

$$Icl^s_{12}(\{x^e\}) = \{x^e\},\ Icl^s_{12}(\{x^e\}^c) = \{x^e\}^c$$

and

$$Dcl^s_{12}(\{x^e\}) = \{x^e\},\ Dcl^s_{12}(\{x^e\}^c) = \{x^e\}^c.$$

Thus $\{x^e\}$ and $\{x^e\}^c$ are IPS (resp. DPS)-soft sets and $X_E = \{x^e\} \sqcup \{x^e\}^c$. So $(X, \tau_1, \tau_2, E, \lesssim)$ is IP (resp. DP)-soft disconnected.

(3) Suppose $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E, G_E\}$ and assume that $(X, \tau_1, \tau_2, E, \lesssim)$ is IP-soft disconnected. Then there exist two non-null soft sets M_E and N_E such that

$$Icl^s_{12}(M_E) \sqcap N_E = Icl^s_{12}(N_E) \sqcap M_E = \widehat{\phi}_E \text{ and } X_E = M_E \sqcup N_E.$$

Thus we have two cases: either G^c_E is an IPC-soft set or not an IPC-soft set.

Case 1. Suppose G^c_E is an IPC-soft set. Then we have three cases.

(i) If $[Icl^s_{12}(M_E) = G^c_E$ and $Icl^s_{12}(N_E) = X_E]$ or $[Icl^s_{12}(N_E) = G^c_E$ and $Icl^s_{12}(M_E) = X_E]$, without loss of generalization, we assume that $Icl^s_{12}(M_E) = G^c_E$ and $Icl^s_{12}(N_E) = X_E$, which a contradiction with disjointness between $Icl^s_{12}(N_E)$ and M_E.

(ii) If $Icl^s_{12}(M_E) = Icl^s_{12}(N_E) = G^c_E$, then it follows that $M_E \sqsubseteq G^c_E$ and $N_E \sqsubseteq G^c_E$ implies $M_E \sqcup N_E \sqsubseteq G^c_E$. Thus $X_E = G^c_E$, a contradicts with that $G_E \neq \widehat{\phi}_E$.

(iii) If $Icl^s_{12}(M_E) \neq G^c_E$ and $Icl^s_{12}(N_E) \neq G^c_E$, then $Icl^s_{12}(M_E) = Icl^s_{12}(N_E) = X_E$, which a contradiction with disjointness between $Icl^s_{12}(M_E)$ and N_E.

Case 2. Suppose G^c_E is not an IPC-soft set. Then $Icl^s_{12}(M_E) = Icl^s_{12}(N_E) = X_E$, which a contradiction with disjointness between $Icl^s_{12}(M_E)$ and N_E.

So in either cases, $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP-soft connected. $(X, \tau_1, \tau_2, E, \lesssim)$ is a DP-soft connected in a similar way. □

Theorem 4.5. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. Then the following are equivalent:*

(1) *$(X, \tau_1, \tau_2, E, \lesssim)$ is IP* (resp. DP)*-soft connected,*

(2) *X_E cannot represented as a union of two non-null disjoint IPO* (resp. DPO)*-soft sets,*

(3) *X_E cannot represented as a union of two non-null disjoint IPC* (resp DPC)*-soft sets,*

(4) *X_E has no proper soft subset which is both IPO* (resp. DPO)*- and IPC* (resp. DPC)*-soft set.*

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds and assume that there exist two non-null $IPO-$soft sets G_E and H_E such that $G_E \sqcap H_E = \widehat{\phi}_E$ and $X_E = G_E \sqcup H_E$. Since $G_E \sqcap H_E = \widehat{\phi}_E$, $G_E \sqsubseteq H^c_E$, $H_E \sqsubseteq G^c_E$. Thus $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_{12}(H_E) \sqcap$

$G_E = \widehat{\phi}_E$. It follows that X_E has an IP-soft separation, i.e., $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP-soft disconnected which contradicts with (1).

(2) $\Rightarrow$ (3): Suppose (2) holds and assume that there exist two non-null $IPC-$soft sets F_E and M_E such that $F_E \sqcap M_E = \widehat{\phi}_E$ and $X_E = F_E \sqcup M_E$. Then F_E^c and M_E^c are non-null DPO-soft sets and $F_E^c \sqcup M_E^c = X_E$, which contradicts with (2).

(3) $\Rightarrow$ (4): Suppose (3) holds and assume that there exists $G_E \sqsubseteq X_E$, $G_E \neq X_E$ and $N_E \neq \widehat{\phi}_E$ such that N_E is both IPO and IPC-soft set. Then N_E and N_E^c are non-null disjoint IPC-soft set and $X_E = N_E \sqcup N_E^c$, which contradicts with (3).

(4) $\Rightarrow$ (1): Suppose (4) holds and assume that $(X, \tau_1, \tau_2, E, \lesssim)$ is IP (resp. DP)-soft disconnected. Then there exist two non-null IPS-soft sets G_E and H_E such that $X_E = G_E \sqcup H_E$. Thus $G_E^c \sqcap H_E^c = \widehat{\phi}_E$ implies $G_E^c \sqsubseteq H_E$, $H_E^c \sqsubseteq G_E$. Since $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$, $Icl^s_{12}(G_E) \sqsubseteq H_E^c \sqsubseteq G_E$. So G_E is an IPC-soft set. Similarly, H_E is IPC-soft set. On the other hand, by Proposition 3.10, we deduce that $G_E \sqsubseteq H_E^c$. Hence $G_E = H_E^c$. It follows that H_E^c is an IPC- soft set. Therefore IPO and IPC-soft set, which contradicts with (4). The proof is similar in case of $(X, \tau_1, \tau_2, E, \lesssim)$ is a DP-soft connected. □

Corollary 4.6. *An SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP* (resp. *DP*)*-soft connected space if and only if the only soft sets over X which are IPO* (resp. *DPO*) *and IPC* (resp. *DPC*)*-soft sets are X_E and $\widehat{\phi}_E$.*

Example 4.7. Let $E = \{e_1, e_2, e_3\}, \lesssim = \blacktriangle \cup \{(a, b)\}$ be a partial order relation on $X = \{a, b, c, d\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G^1_E, G^2_E, G^3_E, G^4_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, H^1_E, H^2_E\}$, where

$G^1_E = \{(e_1, \{a, c\}), (e_2, \{a, b, c\}), (e_3, \{c, d\})\}$, $G^2_E = \{(e_1, \varnothing), (e_2, \{a, c\}), (e_3, \{d\})\}$,
$G^3_E = \{(e_1, \{c\}), (e_2, \{b\}), (e_3, \varnothing)\}$, $G^4_E = \{(e_1, \{c\}), (e_2, \{a, b, c\}), (e_3, \{d\})\}$,
$H^1_E = \{(e_1, \{a, b\}), (e_2, \{a, c\}), (e_3, \{a, d\})\}$, $H^2_E = \{(e_1, \{b\}), (e_2, \{c\}), (e_3, \{a, d\})\}$.

Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. Thus we have

$$\tau_{12} = \{\widehat{\phi}_E, X_E, G^1_E, G^2_E, G^3_E, G^4_E, H^1_E, H^2_E, P^1_E, P^2_E, P^3_E, P^4_E, P^5_E\},$$

where $P^1_E = \{(e_1, \{a, b, c\}), (e_2, \{a, b, c\}), (e_3, \{a, c, d\})\}$,
$P^2_E = \{(e_1, \{b\}), (e_2, \{a, c\}), (e_3, \{a, d\})\}$,
$P^3_E = \{(e_1, \{a, b, c\}), (e_2, \{a, b, c\}), (e_3, \{a, d\})\}$,
$P^4_E = \{(e_1, \{b, c\}), (e_2, \{b, c\}), (e_3, \{a, d\})\}$,
$P^5_E = \{(e_1, \{b, c\}), (e_2, \{a, b, c\}), \{a, d\})\}$.

It is easy to see that the only soft sets over X which are IPO (resp. DPO) and IPC (resp. DPC) -soft sets are X_E and $\widehat{\phi}_E$. So by Corollary 4.6, we deduce that $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP (resp. a DP)-soft connected space.

Theorem 4.8. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. Then the following are equivalent:*

(1) *$(X, \tau_1, \tau_2, E, \lesssim)$ is an IP* (resp. *DP*)*-soft disconnected,*

(2)*X_E can represented as a union of two non-null disjoint IPO* (resp. *DPO*)*-soft sets,*

(3) *X_E can represented as a union of two non-null disjoint IPC* (resp. *DPC*)*-soft sets,*

(4) *X_E has a proper soft subset which is both IPO* (resp. *DPO*) *and IPC* (resp. *DPC*)*-soft set.*

Proof. The proof is similar as Theorem 4.5. □

Remark 4.9. Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an IP (resp. a DP)- soft connected space and let $e \in E$. Then $(X, \tau_1^e, \tau_2^e, \lesssim)$ may not be an IP (resp. a DP)-connected space as shown in the following example.

Example 4.10. Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(b, c)\}$ be a partial order relation on $X = \{a, b, c\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, H_E\}$,
where $G_E = \{(e_1, \{a\}), (e_2, \{b, c\})\}$, $H_E = \{(e_1, \{b, c\}), (e_2, \{a, c\})\}$.
Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. It is clear that $\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E\}$. Thus $(X, \tau_1, \tau_2, E, \lesssim)$ is IP-soft connected and DP-soft connected because we cannot represented X_E as a union of two non-null disjoint IPO-soft sets and DPO-soft sets, respectively. On the other hand, $\tau_{12}^{IPe_1} = \tau_{12}^{DPe_1} = \{\varnothing, X, \{a\}, \{b, c\}\}$. So $(X, \tau_1^{e_1}, \tau_2^{e_1}, \lesssim)$ is an IP-disconnected and a DP-disconnected space because $\{a\}$ is both an IPO (resp. a DPO) and an IPC (resp. a DPC)-soft set.

Remark 4.11. Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an IP (resp. a DP)-soft disconnected space and let $e \in E$. Then $(X, \tau_1^e, \tau_2^e, \lesssim)$ may not be an IP (resp. a DP)-disconnected space as shown in the following example.

Example 4.12. Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(a, c)\}$ be a partial order relation on $X = \{a, b, c\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, M_E, N_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, K_E\}$, where
$M_E = \{(e_1, \{b\}), (e_2, \{b\})\}$, $N_E = \{(e_1, X), (e_2, \{b\})\}$, $K_E = \{(e_1, \varnothing), (e_2, \{a, c\})\}$.
Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. It is clear that $\tau_{12} = \{\widehat{\phi}_E, X_E, M_E, N_E, K_E, P_E\}$, where $P_E = \{(e_1, \{b\}), (e_2, X)\}$. Since, $\{(e_1, X), (e_2, \{b\})\}$ is both an IPO (resp. a DPO) and an IPC (resp. a DPC)-soft set, by Theorem 4.8 (4), $(X, \tau_1, \tau_2, E, \lesssim)$ is IP-soft disconnected and DP-soft disconnected. Now, $\tau_1^{e_1} = \{\varnothing, X, \{b\}\}$ and $\tau_2^{e_1} = \{\varnothing, X\}$. Thus $\tau_{12}^{IPe_1} = \tau_{12}^{DPe_1} = \{\varnothing, X, \{b\}\}$. Obvious that $(X, \tau_1^{e_1}, \tau_2^{e_1}, \lesssim)$ is an IP-connected and a DP-connected space. We can show that $(X, \tau_1^{e_2}, \tau_2^{e_2}, \lesssim)$ is an IP-disconnected and a DP-disconnected space.

Theorem 4.13. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and let $Y \subseteq X$. Then $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ is an IP* (resp. *a DP*)*-soft disconnected if and only if there exist two IPS* (resp. *DPS*)*-soft sets F_E^Y and K_E^Y in $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ such that $Y_E = F_E^Y \sqcup K_E^Y$, where $F_E^Y = Y_E \sqcap F_E$, $K_E^Y = Y_E \sqcap K_E$, F_E, $K_E \in \tau_{12}$.*

Proof. Straightforward. □

Definition 4.14. A property P of an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is called *hereditary property*, if every soft bitopological ordered subspace $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ of $(X, \tau_1, \tau_2, E, \lesssim)$ is also has the property P.

Remark 4.15. The increasing (resp. decreasing) soft connectedness does not hereditary property as shown in the following example.

Example 4.16. From Example 4.5 in [13], if $\lesssim = \blacktriangle \cup \{(y, w)\}$, then we have $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP-soft connected and a DP-soft connected space. But $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ is an IP-soft disconnected and a DP-soft disconnected space. Thus the increasing (resp. decreasing) soft connectedness does not hereditary property.

Theorem 4.17. *Let $(X, \delta_1, \delta_2, E, \lesssim)$ be an SBTOS finer than an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$.*

(1) *If $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP* (resp. *a DP*)*-soft disconnected space, then $(X, \delta_1, \delta_2, E, \lesssim)$ is IP* (resp. *DP*)*-soft disconnected.*

(2) *If $(X, \delta_1, \delta_2, E, \lesssim)$ is an IP* (resp. a *DP*)*-soft connected space, then $(X, \tau_1, \tau_2, E, \lesssim)$ is IP* (resp. *DP*)*-soft connected.*

Proof. (1) Suppose $(X, \delta_1, \delta_2, E, \lesssim)$ is an IP-soft disconnected space. Then there exist $G_E,\ H_E \in P(X)^E$ such that $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E,\ Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$ and $G_E \sqcup H_E = X_E$. Since $(X, \delta_1, \delta_2, E, \lesssim)$ is finer than of $(X, \tau_1, \tau_2, E, \lesssim)$, $\tau_{12} \subseteq \delta_{12}$. It follows that for any soft set G_E, we have $Icl^s_{\delta_{12}}(G_E) \sqsubseteq Icl^s_{\tau_{12}}(G_E)$. Thus we have

$$Icl^s_{\delta_{12}}(G_E) \sqcap H_E = \widehat{\phi}_E,\ Icl^s_{\delta_{12}}(H_E) \sqcap G_E = \widehat{\phi}_E \text{ and } G_E \sqcup H_E = X_E.$$

So $(X, \delta_1, \delta_2, E, \lesssim)$ is an IP-soft disconnected space.

(2) Suppose $(X, \delta_1, \delta_2, E, \lesssim)$ is an IP-soft connected space. Assume that $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP-soft disconnected space. Then by (1), $(X, \delta_1, \delta_2, E, \lesssim)$ is an IP-soft disconnected space, a contradiction.

The proof is similar in case of $(X, \tau_1, \tau_2, E, \lesssim)$ is a DP-soft connected. □

Theorem 4.18. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. If $(X, \tau_1, E, \lesssim)$ or $(X, \tau_2, E, \lesssim)$ is an increasing* (resp. *a* decreasing) *soft disconnected space, then $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP* (resp. *a DP*)*-soft disconnected space.*

Proof. It is immediate from the fact that: $Icl^s_{\tau_{12}}(G_E) = Icl^s_{\tau_1}(G_E) \sqcap Icl^s_{\tau_2}(G_E)$. □

Remark 4.19. Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$. If $(X, \tau_1, E, \lesssim)$ and $(X, \tau_2, E, \lesssim)$ are both increasing (resp. decreasing) soft connected spaces, then $(X, \tau_1, \tau_2, E, \lesssim)$ may not be an IP (resp. a DP)-soft connected space as shown in the following example.

Example 4.20. Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(a, c)\}$ be a partial order relation on $X = \{a, b, c, d\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, M_E\},\ \tau_2 = \{\widehat{\phi}_E, X_E, N_E\}$,
where $M_E = \{(e_1, \{a, c\}), (e_2, \{b, d\})\},\ N_E = \{(e_1, \{b, d\}), (e_2, \{a, c\})\}$.
Then $(X, \tau_1, E, \lesssim)$ and $(X, \tau_2, E, \lesssim)$ are both increasing (resp. decreasing) soft connected spaces. Obvious that $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. Moreover, $\tau_{12} = \{\widehat{\phi}_E, X_E, M_E, N_E\}$. Since M_E and N_E are non-null disjoint IPO (resp. DPO)-soft sets, $M_E \sqcup N_E = X_E$. Then by Theorem 4.8, we deduce that $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP (resp. a DP)-soft disconnected space.

Theorem 4.21. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and let $\varnothing \neq Y \subseteq X$ and let $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ be an IP* (resp. *a DP*)*-soft connected space. If G_E and H_E are an IP* (resp. *a DP*)*-soft separation of X_E, then $Y_E \sqsubseteq G_E$ or $Y_E \sqsubseteq H_E$.*

Proof. Suppose G_E and H_E are an IP-soft separation of X_E and assume that $Y_E \not\sqsubseteq G_E$ and $Y_E \not\sqsubseteq H_E$. Then $Y_E \sqsubseteq X_E = G_E \sqcup H_E$ implies $Y_E \sqcap [G_E \sqcup H_E] = Y_E$. It follows that $[Y_E \sqcap G_E] \sqcup [Y_E \sqcap H_E] = Y_E$. On the other hand, since $Y_E \not\sqsubseteq G_E, Y_E \not\sqsubseteq H_E$ and $Y_E \sqsubseteq [G_E \sqcup H_E]$, $\widehat{\phi}_E \neq Y_E \sqcap G_E \neq Y_E$ and $\widehat{\phi}_E \neq Y_E \sqcap H_E \neq Y_E$. Since $G_E \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_{12Y}(G_E) = Y_E \sqcap Icl^s_{12X}(G_E)$, we have

$$Icl^s_{12Y}[Y_E \sqcap G_E] \sqcap [Y_E \sqcap H_E] = \widehat{\phi}_E$$

and

$$Icl^s_{12Y}[Y_E \sqcap H_E] \sqcap [Y_E \sqcap G_E] = \widehat{\phi}_E.$$

Thus $Y_E \sqcap G_E$ and $Y_E \sqcap H_E$ are an IP-soft separation of Y_E, which contradicts with that $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ is an IP-soft connected space. So our assumption is not true. Hence $Y_E \sqsubseteq G_E$ or $Y_E \sqsubseteq H_E$.

The proof is similar in case of G_E and H_E are a DP-soft separation. □

5. Increasing (decreasing) pairwise soft connected (disconnected) soft sets

Definition 5.1. A soft set G_E in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. *decreasing*) *pairwise disconnected soft set* (briefly, IP (resp. DP)-disconnected soft set), if there exist two non-null IPO (resp. DPO)-soft sets O^1_E, O^1_E such that

$$G_E \sqcap O^1_E \neq \widehat{\phi}_E,\ G_E \sqcap O^2_E \neq \widehat{\phi}_E,\ G_E \sqsubseteq O^1_E \sqcup O^2_E \text{ and } O^1_E \sqcap O^2_E \sqsubseteq G^c_E.$$

In this case, we say that $O^1_E \sqcup O^2_E$ is IP (resp. DP)-*soft disconnected* of G_E. A soft set G_E is called an IP (resp. a DP)-connected soft set, if has no IP (resp. DP)-soft disconnected set.

Example 5.2. From Example 3.17, let $F_E = \{(e_1, \{w\}), (e_2, \{x\})\}$. Take $O^1_E = G_E$, $O^2_E = H^5_E$. It is clear that G_E, H^5_E are IP-soft sets, and $F_E \sqcap O^1_E \neq \widehat{\phi}_E$, $F_E \sqcap O^2_E \neq \widehat{\phi}_E$, $F_E \sqsubseteq O^1_E \sqcup O^2_E = \{(e_1, \{x, w\}), (e_2, \{x, z, w\})\}$ and $O^1_E \sqcap O^2_E = \widehat{\phi}_E \sqsubseteq F^c_E$. Then F_E is an IP-disconnected soft set. If we take $\lesssim = \blacktriangle \cup \{(w, y)\}$, then we can show that $O^1_E = G_E$, $O^2_E = H^5_E$ are DP-soft disconnected of F_E.

Theorem 5.3. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$. Then $x^e \widehat{\in} Icl^s_{12}$ (G_E) if and only if $G_E \sqcap O^{x^e}_E \neq \widehat{\phi}_E$, $\forall O^{x^e}_E \in \tau^{DP}_{12}(x^e)$, where $O^{x^e}_E$ is any DPO-soft set contains x^e and $\tau^{DP}_{12}(x^e)$ is the family of all IPO-soft sets contains x^e.*

Proof. Suppose $x^e \widehat{\in} Icl^s_{12}(G_E)$ and assume that there exists $O^{x^e}_E \in \tau^{DP}_{12}(x^e)$ such that $G_E \sqcap O^{x^e}_E = \widehat{\phi}_E$. Then $G_E \sqsubseteq O^{x^e c}_E$. Thus $Icl^s_{12}(G_E) \sqsubseteq Icl^s_{12}(O^{x^e c}_E) = O^{x^e c}_E$ which implies $Icl^s_{12}(G_E) \sqcap O^{x^e}_E = \widehat{\phi}_E$, a contradiction. Conversely, suppose the necessary condition holds assume that $x^e \widehat{\notin} Icl^s_{12}(G_E)$. Then $x^e \widehat{\in} [Icl^s_{12}(G_E)]^c$. Thus $[Icl^s_{12}(G_E)]^c \in \tau^{DP}_{12}(x^e)$. So by the hypothesis, $[Icl^s_{12}(G_E)]^c \sqcap G_E \neq \widehat{\phi}_E$, a contradiction. □

Theorem 5.4. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$. Then $x^e \widehat{\in} Dcl^s_{12}$ (G_E) if and only if $G_E \sqcap O^{x^e}_E \neq \widehat{\phi}_E$ $\forall O^{x^e}_E \in \tau^{IP}_{12}(x^e)$., where $O^{x^e}_E$ is any IPO-soft set contains x^e and $\tau^{IP}_{12}(x^e)$ is the family of all IPO-soft sets contains x^e.*

Proof. Straightforward. □

Lemma 5.5. *If $O^1_E \sqcup O^2_E$ is an IP* (resp. *DP*)*-soft disconnected of G_E in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$, then $G_E \sqcap O^1_E$ and $G_E \sqcap O^2_E$ are DPS* (resp. *IPS*)*-soft sets.*

Proof. Suppose $O_E^1 \sqcup O_E^2$ is IP-soft disconnected of G_E. Then we have

$$G_E \sqcap O_E^1 \neq \widehat{\phi}_E,\ G_E \sqcap O_E^2 \neq \widehat{\phi}_E,\ G_E \sqsubseteq O_E^1 \sqcup O_E^2 \text{ and } O_E^1 \sqcap O_E^2 \sqsubseteq G_E^c.$$

We shall prove that $G_E \sqcap O_E^1$ and $G_E \sqcap O_E^2$ are DPS-soft sets. Let $x^e \widehat{\in} Dcl_{12}^s(O_E^1 \sqcap G_E)$. Then by Theorem 5.3, $(O_E^1 \sqcap G_E) \sqcap O_E^{x^e} \neq \widehat{\phi}_E\ \forall O_E^{x^e} \widehat{\in} \tau_{12}^{IP}(x^e)$. Now, assume that $x^e \widehat{\in} (O_E^2 \sqcap G_E)$. It follows that $x^e \widehat{\in} O_E^2$. Then $O_E^2 \widehat{\in} \tau_{12}^{IP}(x^e)$. Thus $(O_E^1 \sqcap G_E) \sqcap O_E^2 \neq \widehat{\phi}_E$, which a contradicts with the given $O_E^1 \sqcap O_E^2 \sqsubseteq G_E^c$. So $x^e \not\widehat{\in} (O_E^2 \sqcap G_E)$. Hence $Dcl_{12}^s(O_E^1 \sqcap G_E) \sqcap (O_E^2 \sqcap G_E) = \widehat{\phi}_E$. Similarly, $Dcl_{12}^s(O_E^2 \sqcap G_E) \sqcap (O_E^1 \sqcap G_E) = \widehat{\phi}_E$. Therefore $G_E \sqcap O_E^1$ and $G_E \sqcap O_E^2$ are DPS-soft sets.

The proof is similar in case of $O_E^1 \sqcup O_E^2$ is a DP-soft disconnected sets of G_E. □

Theorem 5.6. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and $G_E \in P(X)^E$. Then $x^e \widehat{\in} Dcl_{12}^s(G_E)$ if and only if $G_E \sqcap O_E^{x^e} \neq \widehat{\phi}_E\ \forall O_E^{x^e} \in \tau_{12}^{DP}(x^e)$, where $O_E^{x^e}$ is any DPO-soft set contains x^e and $\tau_{12}^{DP}(x^e)$ is the family of all DPO-soft sets contains x^e.*

Proof. Straightforward. □

Theorem 5.7. *A soft set G_E in an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is a DP (resp. IP)-disconnected soft set if and only if there exist two IP (resp. DP)-separated soft sets S_E^1, S_E^2 such that $G_E = S_E^1 \sqcup S_E^2$.*

Proof. Suppose that G_E is a DP-disconnected soft set in $(X, \tau_1, \tau_2, E, \lesssim)$. Then G_E has a DP-soft disconnection, say $O_E^1 \sqcup O_E^2$, i.e., there exist two non-null DPO-soft sets O_E^1, O_E^2 such that $G_E \sqcap O_E^1 \neq \widehat{\phi}_E$, $G_E \sqcap O_E^2 \neq \widehat{\phi}_E$, $G_E \sqsubseteq O_E^1 \sqcup O_E^2$ and $O_E^1 \sqcap O_E^2 \sqsubseteq G_E^c$. Then by lemma 5.5, it follows that $G_E \sqcap O_E^1$, $G_E \sqcap O_E^2$ are IPS-soft sets. Since $G_E \sqsubseteq O_E^1 \sqcup O_E^2$, $G_E \sqcap (O_E^1 \sqcup O_E^2) = G_E$ implies $(G_E \sqcap O_E^1) \sqcup (G_E \sqcap O_E^2) = G_E$. Take $S_E^1 = G_E \sqcap O_E^1$ and $S_E^2 = G_E \sqcap O_E^2$.

Conversely, let S_E^1, S_E^2 be two IP-soft sets and let $G_E \in P(X)^E$ such that $G_E = S_E^1 \sqcup S_E^2$. Then $Icl_{12}^s(S_E^1) \sqcap S_E^2 = \widehat{\phi}_E$ and $Icl_{12}^s(S_E^2) \sqcap S_E^1 = \widehat{\phi}_E$. Take $O_E^1 = [Icl_{12}^s(S_E^1)]^c$ and $O_E^2 = [Icl_{12}^s(S_E^2)]^c$. Then O_E^1, O_E^2 are non-null DPO-soft sets. Since $Icl_{12}^s(S_E^1) \sqcap S_E^2 = \widehat{\phi}_E$, $S_E^1 \sqsubseteq [Icl_{12}^s(S_E^2)]^c = O_E^2$. By similar, we also have $S_E^2 \sqsubseteq O_E^1$. It follows that $G_E \sqsubseteq O_E^1 \sqcup O_E^2$. Since $[Icl_{12}^s(S_E^1)]^c \sqsubseteq S_E^{1c}$, $[Icl_{12}^s(S_E^2)]^c \sqsubseteq S_E^{2c}$, $O_E^1 \sqcap O_E^2 \sqsubseteq G_E^c$. Furthermore, since S_E^1, $S_E^2 \sqsubseteq G_E$ and $S_E^2 \sqsubseteq O_E^1$, $S_E^1 \sqsubseteq O_E^2$, $S_E^2 \sqsubseteq G_E \sqcap O_E^1$ and $S_E^1 \sqsubseteq G_E \sqcap O_E^2$. But $S_E^1 \neq \widehat{\phi}_E$, $S_E^2 \neq \widehat{\phi}_E$. Thus $G_E \sqcap O_E^1 \neq \widehat{\phi}_E$, $G_E \sqcap O_E^2 \neq \widehat{\phi}_E$. So G_E is a DP-disconnected soft set.

The proof is similar in case of G_E is an IP-disconnected soft set. □

Corollary 5.8. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. If S_E^1, S_E^2 are two IPS (resp. DPS)-soft sets, then $S_E^1 \sqcup S_E^2$ is a DP (resp. IP)-disconnected soft set.*

Corollary 5.9. *A soft set G_E in an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is IP (resp. DP)-connected soft set if and only if it cannot expressed as a union of two IDS (resp. IPS)-soft sets.*

Proposition 5.10. *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS.*

1. *Every soft point is an IP (resp. a DP)-connected soft set.*
2. *The null soft set is an IP (resp. a DP)-connected soft set.*

Proof. (1) Let $x^e \widehat{\in} X_E$. Then for any two non-null IPO-soft sets O_E^1, O_E^2 such that $\{x^e\} \sqcap O_E^1 \neq \widehat{\phi}_E$, $\{x^e\} \sqcap O_E^2 \neq \widehat{\phi}_E$, we have $x^e \widehat{\in} O_E^1 \sqcap O_E^2$. It follows that $O_E^1 \sqcap O_E^2 \not\sqsubseteq \{x^e\}^c$. Thus x^e is an IP-connected soft set.

The proof is similar in case of a DP-connected soft set.

(2) Obvious. □

Theorem 5.11. *Let F_E be an IP* (resp. *DP*)*-connected soft set in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ and let $F_E \sqsubseteq M_E \sqsubseteq Icl_{12}^s(F_E)$. Then M_E, $Icl_{12}^s(F_E)$ are also IP* (resp. *DP*)*-connected soft sets.*

Proof. Let F_E be an IP-connected soft set in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ and assume that M_E is an IP-disconnected soft set in an $(X, \tau_1, \tau_2, E, \lesssim)$. Then there exist two non-null IPO-soft sets O_E^1, O_E^2 such that

$$M_E \sqcap O_E^1 \neq \widehat{\phi}_E,\ M_E \sqcap O_E^2 \neq \widehat{\phi}_E,\ M_E \sqsubseteq O_E^1 \sqcup O_E^2$$

and

$$O_E^1 \sqcap O_E^2 \sqsubseteq M_E^c.$$

Since $F_E \sqsubseteq M_E$, $F_E \sqsubseteq O_E^1 \sqcup O_E^2$ and $O_E^1 \sqcap O_E^2 \sqsubseteq F_E$. Since F_E is an IP-connected soft set, either $F_E \sqcap O_E^1 = \widehat{\phi}_E$ or $F_E \sqcap O_E^2 = \widehat{\phi}_E$. If we claim that $F_E \sqcap O_E^1 = \widehat{\phi}_E$, then O_E^{1c} is a DPO-soft set contains F_E. It follows that $Icl_{12}^s(F_E) \sqsubseteq O_E^{1c}$ which implies that $M_E \sqcap O_E^1 = \widehat{\phi}_E$, a contradicts with our assumption. Thus our assumption is false. So M_E is an IP-connected soft set. In particular, put $M_E = Icl_{12}^s(F_E)$. Then $Icl_{12}^s(F_E)$ is also an IP-connected soft set.

The proof is similar in case of F_E is a DP-disconnected soft set. □

Remark 5.12. The soft subset of an IP (resp. DP)-soft connected space need not be an IP (resp. DP)-connected soft set as seen in the following example.

Example 5.13. Consider Example 4.7. Let $\lesssim = \blacktriangle \cup \{(a, c)\}$. Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an IP-soft connected space. Now, let $F_E = \{(e_1, \{c\}), (e_2, \varnothing), (e_3, \{d\})\}$. Take $O_E^1 = G_E^2$, $O_E^2 = G_E^3$. Then we have

$$F_E \sqcap O_E^1 = \{(e_1, \varnothing), (e_2, \varnothing), (e_2, \{d\})\}, F_E \sqcap O_E^2 = \{(e_1, \{c\}), (e_2, \varnothing), (e_2, \varnothing)\},$$

$$F_E \sqsubseteq O_E^1 \sqcup O_E^2 = \{(e_1, \{c\}), (e_2, \{a, c\}), (e_2, \{d\})\} \text{ and } O_E^1 \sqcap O_E^2 = \widehat{\phi}_E \sqsubseteq F_E^c.$$

Thus F_E is an IP-disconnected soft subset of $(X, \tau_1, \tau_2, E, \lesssim)$. If we take $\lesssim = \blacktriangle \cup \{(c, a)\}$, then we have F_E is a DP-disconnected soft subset of $(X, \tau_1, \tau_2, E, \lesssim)$.

Remark 5.14. The union of two IP (resp. DP)-connected soft sets need not be an IP (resp. DP)-connected soft set as seen in the following example.

Example 5.15. From Example 4.7 and Example 5.13, it is clear by Proposition 5.10 (1) that c^{e_1}, d^{e_3} are IP (resp. DP)-connected soft sets. Nevertheless, $\{c^{e_2}\} \sqcup \{d^{e_3}\} = \{(e_1, \{c\}), (e_2, \varnothing), (e_2, \{d\})\} = F_E$ is an IP-disconnected soft set and a DP-disconnected soft set.

Theorem 5.16. *Let G_E, H_E be two IP* (resp. *DP*)*-connected soft sets in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$. If $G_E \sqcap H_E \neq \widehat{\phi}_E$, then $G_E \sqcup H_E$ is an IP* (resp. *DP*)*-connected soft set.*

Proof. Let G_E and H_E be IP (resp. DP)-connected soft sets and suppose $G_E \sqcap H_E \neq \widehat{\phi}_E$. Assume that $G_E \sqcup H_E$ is an IP-disconnected soft set. Then there exist two non-null IPO-soft sets O^1_E, O^2_E such that $[G_E \sqcup H_E] \sqcap O^1_E \neq \widehat{\phi}_E$, $[G_E \sqcup H_E] \sqcap O^2_E \neq \widehat{\phi}_E$, $[G_E \sqcup H_E] \sqsubseteq O^1_E \sqcup O^2_E$ and $[O^1_E \sqcap O^2_E] \sqsubseteq [G_E \sqcup H_E]^c$. Since $G_E \sqsubseteq G_E \sqcup H_E$, $G_E[\sqsubseteq O^1_E \sqcup O^2_E]$ and $O^1_E \sqcap O^2_E \sqsubseteq G^c_E$. Since G_E is an IP-connected soft set, $G_E \sqcap O^1_E = \widehat{\phi}_E$ or $G_E \sqcap O^2_E = \widehat{\phi}_E$. Thus $G_E \sqsubseteq O^1_E$ or $G_E \sqsubseteq O^2_E$ for $[G_E \sqsubseteq O^1_E \sqcup O^2_E]$. Similarly, $H_E \sqsubseteq O^1_E$ or $H_E \sqsubseteq O^2_E$. Thus, if $G_E \sqsubseteq O^1_E$ and $H_E \sqsubseteq O^2_E$, then $G_E \sqcap H_E \sqsubseteq O^1_E \sqcap O^2_E \sqsubseteq G^c_E \sqcap H^c_E$ which implies that $G_E \sqcap H_E = \widehat{\phi}_E$, a contradiction. Similarly, when $G_E \sqsubseteq O^2_E$ and $H_E \sqsubseteq O^1_E$, we have a contradiction. So our assumption is not true. Hence $G_E \sqcup H_E$ is an IP-connected soft set. □

Theorem 5.17. *Let ϕ_ψ be an ISP (resp. a DSP)-continuous and injective mapping from $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim_1)$ in to an $SBTOS$ $(Y, \eta_1, \eta_2, K, \lesssim_2)$. If G_E is an IP (resp DP)-connected soft set in $(X, \tau_1, \tau_2, E, \lesssim_1)$, then $\phi_\psi(G_E)$ is an IP (resp. a DP)-connected soft set in $(Y, \eta_1, \eta_2, K, \lesssim_2)$.*

Proof. Suppose G_E is an IP-connected soft set in $(X, \tau_1, \tau_2, E, \lesssim_1)$. Assume that $\phi_\psi(G_E)$ is not an IP-connected soft set in $(Y, \eta_1, \eta_2, K, \lesssim_2)$. Then there exist two non-null IPO-soft sets O^1_K, O^2_K such that $\phi_\psi(G_E) \sqcap O^1_K \neq \widehat{\phi}_E$, $\phi_\psi(G_E) \sqcap O^2_K \neq \widehat{\phi}_E$, $\phi_\psi(G_E) \sqsubseteq O^1_K \sqcup O^2_K$ and $[O^1_K \sqcap O^2_K] \sqsubseteq Y_K - \phi_\psi(G_E)$. Then by Theorem 4.8 and Proposition 2.8, it follows that $G_E \sqcap \phi^{-1}_\psi(O^1_K) \neq \widehat{\phi}_E$, $G_E \sqcap \phi^{-1}_\psi(O^2_K) \neq \widehat{\phi}_E$, $G_E \sqsubseteq \phi^{-1}_\psi(O^1_K) \sqcup \phi^{-1}_\psi(O^2_K)$ and $[\phi^{-1}_\psi(O^1_K) \sqcap \phi^{-1}_\psi(O^2_K)] \sqsubseteq \phi^{-1}_\psi(Y_K - \phi_\psi(G_E)) = X_K - G_E$. Since ϕ_ψ is an ISP-continuous, $\phi^{-1}_\psi(O^1_K)$, $\phi^{-1}_\psi(O^2_K)$ are IPO-soft sets in $(X, \tau_1, \tau_2, E, \lesssim_1)$. Thus $\phi^{-1}_\psi(O^1_K) \sqcup \phi^{-1}_\psi(O^2_K)$ form an IP-soft disconnection of G_E which contrary to the fact that G_E is an IP-connected soft set in $(X, \tau_1, \tau_2, E, \lesssim_1)$. So $\phi_\psi(G_E)$ is an IP-connected soft set in $(Y, \eta_1, \eta_2, K, \lesssim_2)$. □

In Theorems 5.16 and 5.17, a similar proof can be given for the case between parentheses.

Corollary 5.18. *Let ϕ_ψ be an ISP (resp. a DSP)-continuous and injective mapping from an IP (resp. a DP)-connected soft space $(X, \tau_1, \tau_2, E, \lesssim_1)$ on to an $SBTOS$ $(Y, \eta_1, \eta_2, K, \lesssim_2)$, then $(Y, \eta_1, \eta_2, K, \lesssim_2)$ is an IP (resp. a DP)-soft connected space.*

6. Conclusion

In 1965, Nachbin [1] introduced the concept of topological ordered space, which combines the properties of partial order relations and topological spaces. Later, in 1999, Molodtsov [3] proposed the idea of "soft sets" to address issues related to uncertainty, vagueness, imprecision, and incomplete data.Ittanagi [11] introduced the notion of a soft bitopological space. Building upon these concepts, El-Sheikh et al. [16] introduced the concept of soft bitopological ordered spaces

In this paper, we introduced and studied the notion of IPS (resp. DPS)-soft sets. Based on this notion, we defined and studied some properties and characterizations of IP (resp. DP)-soft connected spaces and IP (resp. DP)-connected soft sets in soft bitopological ordered spaces. Some properties of such notions are obtained. We expect that the findings in this paper can be promoted to the further study on soft bitopology ordered to carry out general framework for the practical life applications.

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